

**PROMPT #1:**

A precalculus class was learning how to solve rational equations. They first solved some simple equations by hand. Then the teacher asked the class to solve the following rational equation using a CAS.

$$\frac{b+2}{b-6} + \frac{2}{b-1} = \frac{-4}{b^2-7b+6}$$

After the equation was entered, the CAS returned the following output.

The screenshot shows a CAS interface with the following content:

$$\frac{b+2}{b-6} + \frac{2}{b-1} = \frac{-4}{b^2-7b+6}$$

$$\frac{2}{b-1} + \frac{8}{b-6} + 1 = \frac{-4}{b^2-7b+6}$$

Below the equations, a warning message is displayed: **⚠ Domain of the result might be larger tha...**

The teacher was perplexed. Why did the CAS give such a strange result accompanied by a domain warning?

**COMMENTARY:**

This example illustrates that the use of technology, particularly computer algebra systems, in the mathematics classroom may produce unexpected outputs and prompt teachers to acquire “mathematical technology content” not only regarding notation but also equivalent forms. Additionally, knowledge of the operation of the technology (i.e. its internal mathematical programming) is helpful in order to understand the reason for the outputs. Thus, the use of technology can alter the content that is taught in a mathematics classroom. It can act as a new content resource for the teacher and students.

**MATHEMATICAL FOCUS #1:** Equivalent forms; Restricted domains; Division by zero

In comparing the two equations, the right side of each is the same. Only the left side of the original equation was altered. Using the Commutative Property of Addition, the change on the left side appears to have occurred only with the expression  $\frac{b+2}{b-6}$ .

When rewriting the quotient in lowest terms, there exists the possibility that a common factor would be eliminated. This might then affect the domain of the variable for which an equation or function was being considered. Since division by zero is undefined, restrictions on the variable may result. For example,  $f(x) = \frac{x-2}{x^2-5x+6}$  would have a restricted domain  $\{x : x \in \mathfrak{R}, x \neq 2, 3\}$ . When simplified,  $f(x) = \frac{x-2}{x^2-5x+6} = \frac{1}{x-3}$ , and the expression now appears to have only  $x = 3$  as a restriction. In lowest terms, one of the factors of the divisor is eliminated, resulting in a domain that included other values not previously restricted. So, the domain got “larger”. A quotient of polynomials is rewritten in the example of Prompt #1. Thus, the warning on the calculator appeared. Interestingly

enough, the polynomial division occurred but no change in divisor resulted.  $\frac{b+2}{b-6}$

became  $\frac{8}{b-6} + 1$ , having the same denominator. But the calculator was not able to determine that the equations were equivalent. So, the key words in the warning are “might be.” Are the equations equivalent? Equations related by reversible steps are called equivalent equations. Thus, the equations are indeed equivalent.

But why is  $\frac{b+2}{b-6} = \frac{8}{b-6} + 1$ ?

### MATHEMATICAL FOCUS #2: Division of Polynomials; Quotient-Remainder Theorem for Polynomials

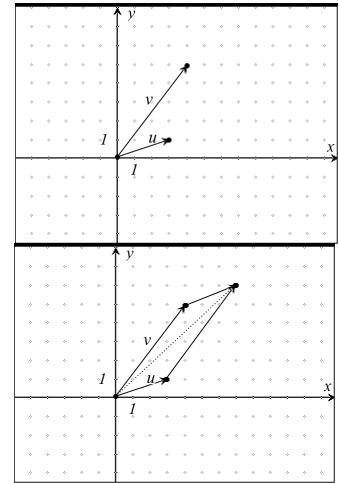
The Quotient-Remainder Theorem for Polynomials states that if  $n(x)$  is a polynomial and  $d(x)$  is a nonzero polynomial, then there exist unique polynomials  $q(x)$  and  $r(x)$  such that for all real numbers  $x$ , (1)  $n(x) = q(x) \cdot d(x) + r(x)$  and (2) either degree of  $r(x) <$  degree of  $d(x)$  or  $r(x)$  is the zero polynomial. The polynomial  $q(x)$  is called the quotient and  $r(x)$  is called the remainder for the division of  $n(x)$  by  $d(x)$ . Using polynomial division,  $b+2 = (b-6) \cdot 1 + 8$ . Dividing both sides of the equation by  $b-6$  results in  $\frac{b+2}{b-6} = \frac{8}{b-6} + 1$ .

### MATHEMATICAL FOCUS #3: Operation of a CAS; programmed to simplify mathematical sentences

The question still remains as to why the CAS produced the unexpected output. Knowledge of the internal mechanism of the technology is sometimes essential when considering the reason for an output. To simplify an expression, some CAS require use of a command such as EXPAND or PROPFRAC. However, these same CAS may be internally programmed to automatically simplify (without specifying commands) sides of a mathematical sentence, as is the case in Prompt #1.  $\frac{2}{b-1}$  is indeed the result of the polynomial division with 0 as the quotient and 2 as the remainder. On the other hand,  $\frac{b+2}{b-6}$  results in a nonzero quotient of 1 with a remainder of 8.

**PROMPT #2:**

In a precalculus class, the teacher was presenting a second example of adding two vectors given in polar form to get the resultant vector. The teacher proceeded to draw the vectors geometrically (as shown at the right) and to convert each vector from polar form to component form in order to add the vectors component-wise, and then to convert the resultant vector back to polar form to find its magnitude. This process was long and had been done in the immediately previous classroom example. A student stopped the teacher after the resultant vector was drawn on the board to ask “Can we solve the problem another way? I know that we haven’t mentioned the Law of Cosines this year, but I think that this problem can be solved with it.”

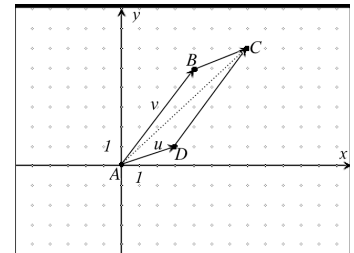


COMMENTARY: This example provides an illustration of how mathematical content knowledge is broadened by multiple representations and alternative solutions.

**MATHEMATICAL FOCUS #1: Angle addition**

Consider angle  $BAD$  of parallelogram  $ABCD$ . Since vectors  $u$  and  $v$  were given in polar form, suppose  $u = [r_1, \theta_1]$  and  $v = [r_2, \theta_2]$ . The measure of angle  $BAD$  can be found using the Angle Addition Theorem.

$$m\angle BAD = \theta_2 - \theta_1.$$



**MATHEMATICAL FOCUS #2: Properties of Parallelograms**

In a parallelogram, opposite sides are congruent so  $CD = r_2$ , and  $BC = r_1$ . Because opposite angles in a parallelogram are congruent and the sum of the measures of the interior angles is 360 degrees, consecutive angles are supplementary. Thus,

$$m\angle ABC = 180 - (\theta_2 - \theta_1).$$

**MATHEMATICAL FOCUS #3: Criteria for using the Law of Cosines**

Are the criteria met for using the Law of Cosines? Either the SAS or SSS conditions must occur. Since angle  $ABC$  is included between sides  $AB$  and  $BC$ , SAS is fulfilled. So the Law of Cosines can be applied.

It is worth noting that the use of the Law of Cosines may be faster for finding the magnitude of the resultant vector, but it will take some work to find the direction of the resultant vector.